Tutorial 4

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5th week

1. Find the first moment about the y-axis of a thin plate of density $\delta(x, y) = 1$ covering the infinite region under the curve $y = e^{-x^2/2}$ in the first quadrant.

$$M_y = \int_0^\infty \int_0^{e^{-x^2/2}} x\,\delta(x,y)\,dy\,dx$$

$$= \int_0^\infty \int_0^{e^{-x^2/2}} x\,dy\,dx$$

$$= \int_0^\infty x\,e^{-x^2/2}\,dy$$

$$= \int_0^\infty e^{-x^2/2}\,d\frac{x^2}{2} \qquad \left(\text{Let } t = \frac{x^2}{2}\right)$$

$$= \int_0^\infty e^{-t}\,dt$$

$$= -e^{-t}\Big|_0^\infty$$

$$= 1$$

2. (a) Solve the system

$$u = x - y,\ v = 2x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

(b) Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy-plane. Sketch the transformed region in the uv-plane.

(a)

$$\begin{cases} u = x - y \\ v = 2x + y \end{cases} \Rightarrow \begin{cases} x = \dfrac{u+v}{3} \\ y = \dfrac{-2u+v}{3} \end{cases}$$

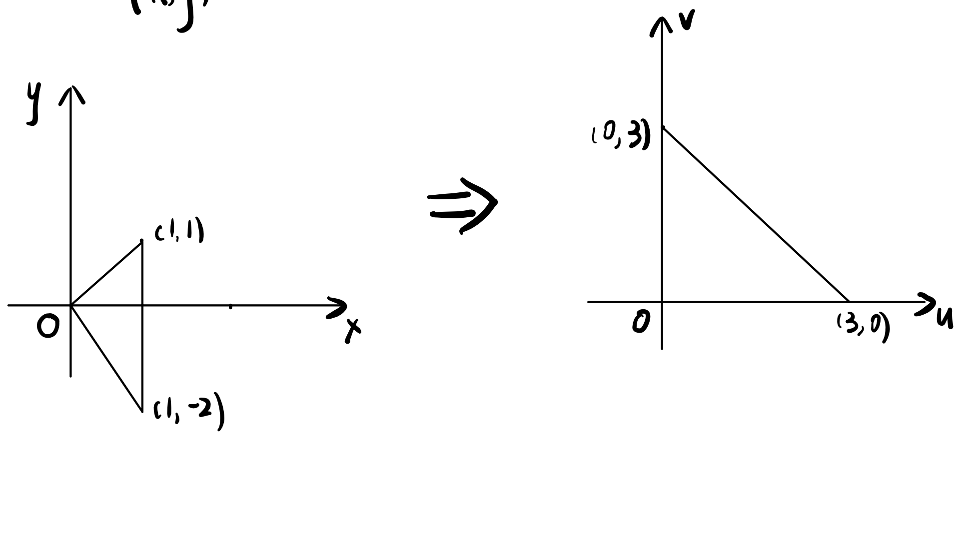
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

(b)

$(x,y) = (0,0) \Rightarrow (u,v) = (0,0)$

$(x,y) = (1,1) \Rightarrow (u,v) = (0,3)$

$(x,y) = (1,-2) \Rightarrow (u,v) = (3,0)$

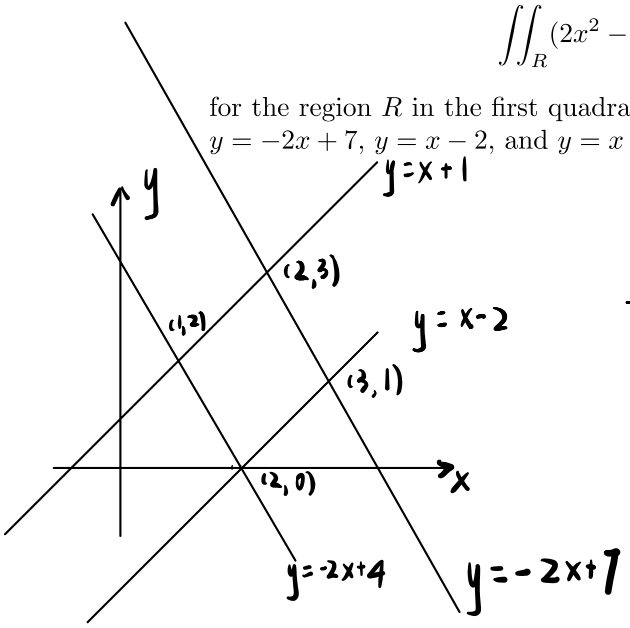


3. Use the transformation in Ex. 2 to evaluate the integral

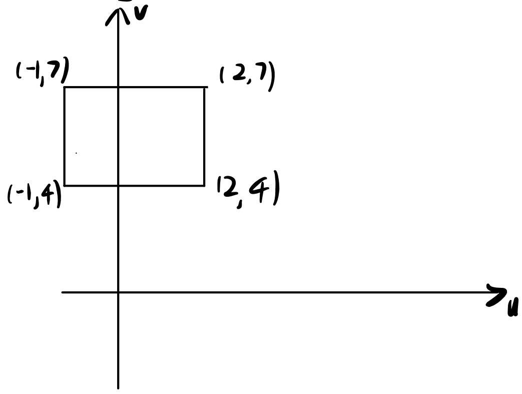
$$\iint_R (2x^2 - xy - y^2)\mathrm{d}x\mathrm{d}y$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

$u = x - y \quad v = 2x + y$



$\Rightarrow$



$(x,y) = (1,2) \Rightarrow (u,v) = (-1,4)$

$(x,y) = (2,0) \Rightarrow (u,v) = (2,4)$

$(x,y) = (2,3) \Rightarrow (u,v) = (-1,7)$

$(x,y) = (3,1) \Rightarrow (u,v) = (2,7)$

$$\iint_R (2x^2 - xy - y^2)\,dx\,dy = \iint_R (2x+y)(x-y)\,dx\,dy$$

$$= \int_{-1}^{2}\int_{4}^{7} uv \frac{\partial(x,y)}{\partial(u,v)}\,dv\,du$$

$$= \frac{1}{3}\int_{-1}^{2}\int_{4}^{7} uv\,dv\,du$$

$$= \frac{1}{3}\int_{-1}^{2} \frac{1}{2}uv^2\Big|_{v=4}^{v=7}\,du$$

$$= \frac{1}{6}\int_{-1}^{2} 33u\,du$$

$$= \frac{11}{2}\cdot\frac{1}{2}u^2\Big|_{-1}^{2}$$

$$= \frac{33}{4}$$

4. Assuming the result that the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base toward the top, show, by transforming the appropriate integrals, that the center of mass of a solid semiellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1, \quad z \geq 0$, lies on the z-axis three-eighths of the way from the base toward the top.

By what we have know

$$\iiint_{\{x^2+y^2+z^2\leq 1,\ z\geq 0\}} z \, dxdydz = \frac{3}{8}\iiint_{\{x^2+y^2+z^2\leq 1,\ z\geq 0\}} 1 \, dxdydz$$

Let $u = \frac{x}{a}$ $\quad v = \frac{y}{b}$ $\quad w = \frac{z}{c}$

We have $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

Similarly $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{abc}$

Hence

$$M_{xy} = \iiint_{\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1,\ z\geq 0\}} z \, dxdydz$$

$$= \iiint_{\{u^2+v^2+w^2\leq 1,\ w\geq 0\}} cw \frac{\partial(x,y,z)}{\partial(u,v,w)} \, dudvdw$$

$$= abc^2 \iiint_{\{u^2+v^2+w^2\leq 1,\ w\geq 0\}} w \, dudvdw$$

$$= abc^2 \frac{3}{8}\iiint_{\{u^2+v^2+w^2\leq 1,\ w\geq 0\}} 1 \, dudvdw$$

$$= abc^2 \frac{3}{8}\iiint_{\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1,\ z\geq 0\}} 1 \frac{\partial(u,v,w)}{\partial(x,y,z)} \, dxdydz$$

$$= \frac{3}{8}c \iiint_{\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1,\ z\geq 0\}} 1 \, dxdydz$$

So $\bar{z} = \frac{3}{8}c$

By symmetry, $\bar{x} = \bar{y} = 0$. Hence the center is $(0, 0, \frac{3}{8})$

5. In previous courses (ex. MATH 1010, MATH 1050), we learned how to find the volume of a solid of revolution using the shell method; namely, if the region between the curve $y = f(x)$ and the x-axis from a to b $(0 < a < b)$ is revolved about the y-axis, the volume of the resulting solid is $\int_a^b 2\pi x f(x)dx$. Prove that finding volumes by using triple integrals gives the same result.

$$D = \left\{ (x, y, z) : \ a \le \sqrt{x^2+y^2} \le b, \quad 0 \le z \le f(\sqrt{x^2+y^2}) \right\}$$

Let $\quad x = r\cos\theta \quad y = r\sin\theta \quad r \in [0, b] \quad \theta \in [0, 2\pi]$

Then we have

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$V = \iiint_D 1 \, dx\,dy\,dz$$

$$= \int_a^b \int_0^{2\pi} \int_0^{f(r)} r \, dz \, d\theta \, dr$$

$$= \int_a^b \int_0^{2\pi} r f(r) \, d\theta \, dr$$

$$= \int_a^b 2\pi r f(r) dr$$

$$= \int_a^b 2\pi x f(x) dx$$